2021

AP CollegeBoard

AP® Calculus BC

Sample Student Responses and Scoring Commentary

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Free Response Question 1

- ☑ Scoring Guideline
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Part A (AB or BC): Graphing calculator required

Question 1 — 9 points

General Scoring Notes

Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be correct to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.

Scoring guidelines and notes contain examples of the most common approaches seen in student responses. These guidelines can be applied to alternate approaches to ensure that these alternate approaches are scored appropriately.

r (centimeters)	0	1	2	2.5	4
$f(r)$ (milligrams per square centimeter)	1	2	6	10	18

The density of a bacteria population in a circular petri dish at a distance r centimeters from the center of the dish is given by an increasing, differentiable function f, where $f(r)$ is measured in milligrams per square centimeter. Values of $f(r)$ for selected values of r are given in the table above.

Model Solution	Scoring	
(a) Use the data in the table to estimate $f'(2.25)$. Using correct units, interpret the meaning of your answer in the context of this problem.		
$f'(2.25) \approx \dfrac{f(2.5) - f(2)}{2.5 - 2} = \dfrac{10 - 6}{0.5} = 8$	Estimate	**1 point**
At a distance of $r = 2.25$ centimeters from the center of the petri dish, the density of the bacteria population is increasing at a rate of 8 milligrams per square centimeter per centimeter.	Interpretation with units	**1 point**

Scoring notes:

- To earn the first point the response must provide both a difference and a quotient and must explicitly use values of f from the table.
- Simplification of the numerical value is not required to earn the first point. If the numerical value is simplified, it must be correct.
- The interpretation requires all of the following: distance $r = 2.25,$ density of bacteria (population) is increasing or changing, at a rate of $8,$ and units of milligrams per square centimeter per centimeter.
- The second point (interpretation) cannot be earned without a nonzero presented value for $f'(2.25)$.
- To earn the second point the interpretation must be consistent with the presented nonzero value for $f'(2.25)$. In particular, if the presented value for $f'(2.25)$ is negative, the interpretation must include "decreasing at a rate of $|f'(2.25)|$" or "changing at a rate of $f'(2.25)$." The second point cannot be earned for an incorrect statement such as "the bacteria density is decreasing at a rate of -8 … " even for a presented $f'(2.25) = -8$.
- The units ($\text{mg/cm}^2\text{/cm}$) may be attached to the estimate of $f'(2.25)$ and, if so, do not need to be repeated in the interpretation.
- If units attached to the estimate do not agree with units in the interpretation, read the units in the interpretation.

Total for part (a) 2 points

(b) The total mass, in milligrams, of bacteria in the petri dish is given by the integral expression $2\pi\int_0^4 r\,f(r)\,dr$. Approximate the value of $2\pi\int_0^4 r\,f(r)\,dr$ using a right Riemann sum with the four subintervals indicated by the data in the table.

$2\pi\int_0^4 r\,f(r)\,dr \approx 2\pi(1\cdot f(1)\cdot(1-0)+2\cdot f(2)\cdot(2-1)$ $+\,2.5\cdot f(2.5)\cdot(2.5-2)+4\cdot f(4)\cdot(4-2.5))$	Right Riemann sum setup	**1 point**
$=2\pi(1\cdot 2\cdot 1+2\cdot 6\cdot 1+2.5\cdot 10\cdot 0.5+4\cdot 18\cdot 1.5)$ $=269\pi=845.088$	Approximation	**1 point**

Scoring notes:

- The presence or absence of 2π has no bearing on earning the first point.
- The first point is earned for a sum of four products with at most one error in any single value among the four products. Multiplication by 1 in any term does not need to be shown, but all other products must be explicitly shown.
- A response of $1\cdot f(1)\cdot(1-0)+2\cdot f(2)\cdot(2-1)+2.5\cdot f(2.5)\cdot(2.5-2)+4\cdot f(4)\cdot(4-2.5)$ earns the first point but not the second.
- A response with any error in the Riemann sum is not eligible for the second point.
- A response that provides a completely correct left Riemann sum for $2\pi\int_0^4 r\,f(r)\,dr$ and approximation (91π) earns one of the two points. A response that has any error in a left Riemann sum or evaluation for $2\pi\int_0^4 r\,f(r)\,dr$ earns no points.
- A response that provides a completely correct right Riemann sum for $2\pi\int_0^4 f(r)\,dr$ and approximation (80π) earns one of the two points. A response that has any error in a right Riemann sum or evaluation for $2\pi\int_0^4 f(r)\,dr$ earns no points.
- Simplification of the numerical value is not required to earn the second point. If a numerical value is given, it must be correct to three decimal places.

Total for part (b) 2 points

(c) Is the approximation found in part (b) an overestimate or underestimate of the total mass of bacteria in the petri dish? Explain your reasoning.

$\frac{d}{dr}(r\,f(r)) = f(r) + r\,f'(r)$	Product rule expression for $\frac{d}{dr}(r\,f(r))$	**1 point**
Because f is nonnegative and increasing, $\frac{d}{dr}(r\,f(r)) > 0$ on the interval $0 \le r \le 4$. Thus, the integrand $r\,f(r)$ is strictly increasing. Therefore, the right Riemann sum approximation of $2\pi\int_0^4 r\,f(r)\,dr$ is an overestimate.	Answer with explanation	**1 point**

Scoring notes:

- To earn the second point a response must explain that $r\,f(r)$ is increasing and, therefore, the right Riemann sum is an overestimate. The second point can be earned without having earned the first point.
- A response that attempts to explain based on a left Riemann sum for $2\pi\int_0^4 r\,f(r)\,dr$ from part (b) earns no points.
- A response that attempts to explain based on a right Riemann sum for $2\pi\int_0^4 f(r)\,dr$ from part (b) earns no points.

Total for part (c) 2 points

(d) The density of bacteria in the petri dish, for $1 \le r \le 4$, is modeled by the function g defined by $g(r) = 2 - 16\left(\cos\left(1.57\sqrt{r}\right)\right)^3$. For what value of k, $1 < k < 4$, is $g(k)$ equal to the average value of $g(r)$ on the interval $1 \le r \le 4$?

Average value $= g_{\text{avg}} = \dfrac{1}{4-1}\displaystyle\int_1^4 g(r)\,dr$	Definite integral	**1 point**
$\dfrac{1}{4-1}\displaystyle\int_1^4 g(r)\,dr = 9.875795$	Average value	**1 point**
$g(k) = g_{\text{avg}} \Rightarrow k = 2.497$	Answer	**1 point**

Scoring notes:

- The first point is earned for a definite integral, with or without $\dfrac{1}{4-1}$ or $\dfrac{1}{3}$.
- A response that presents a definite integral with incorrect limits but a correct integrand earns the first point.
- Presentation of the numerical value 9.875795 is not required to earn the second point. This point can be earned by the average value setup: $\dfrac{1}{3}\displaystyle\int_1^4 g(r)\,dr$.
- Once earned for the average value setup, the second point cannot be lost. Subsequent errors will result in not earning the third point.
- The third point is earned only for the value $k = 2.497$.
- The third point cannot be earned without the second.
- Special case: A response that does not provide the average value setup but presents an average value of -13.955 is using degree mode on their calculator. This response would not earn the second point but could earn the third point for an answer of $k = 2.5$ (or 2.499).

Total for part (d) 3 points

Total for question 1 9 points

Sample 1A

Answer QUESTION 1 parts (a) and (b) on this page.

r (centimeters)	0	1	2	2.5	4
$f(r)$ (milligrams per square centimeter)	1	2	6	10	18

Response for question 1(a)

$f'(2.25) \approx \dfrac{f(2.5)-f(2)}{2.5-2} = \dfrac{10-6}{0.5} = \dfrac{4}{0.5} = 8$ milligrams per square centimeter per centimeter

The density of bacteria changes at a rate of approximately 8 milligrams per square centimeter per centimeter at distance $r=2.25$ centimeters from the center of the dish

Response for question 1(b)

$2\pi\int_0^4 r\cdot f(r)\,dr \approx 2\pi(2\cdot1\cdot1 + 6\cdot1\cdot2 + 10\cdot0.5\cdot2.5 + 18\cdot1.5\cdot4)$

$= 2\pi(2+12+12.5+108) = 2\pi(134.5) = 269\pi$ milligrams

Answer QUESTION 1 parts (c) and (d) on this page.

Response for question 1(c)

As a rule, Right riemann Sums are always an over estimate for functions with possitive slope, and underestimates for functions with negative slope. The slope of $r \cdot f(r)$ is equal to $r' \cdot f(r) + r \cdot f'(r)$. Since r, r', $f(r)$, and $f'(r)$ are always positive on the interval $[0,4]$, $r \cdot f(r)$ always has a positive slope on that interval. Since it's a positive sloped function, the right riemann sum for $r \cdot f(r)$ from 0 to 4 is an over estimate.

Response for question 1(d)

$$\text{Avg of } g(r) = \frac{1}{4-1}\int_1^4 \left(2-16(\cos(1.57\sqrt{r}))^3\right) dr = \frac{29.627}{3} = 9.876$$

$$g(k) = 2-16(\cos(1.57\sqrt{k}))^3 = 9.876$$

$$k = 2.497$$

Answer QUESTION 1 parts (a) and (b) on this page.

r (centimeters)	0	1	2	2.5	4
$f(r)$ (milligrams per square centimeter)	1	2	6	10	18

Response for question 1(a)

$f'(2.25) \approx \frac{f(2.5)-f(2)}{2.5-2} = \frac{10-6}{0.5} = 8 \ \frac{\text{milligrams}}{\text{cm}^3}$

The rate of change of $f(r)$ at distance $r = 2.25$ centimeters is approximately 8 milligrams per cubic centimeter

Response for question 1(b)

$2\pi \int_0^4 r f(r)\,dr \approx 2\pi[1(1)f(1) + 1(2)f(2) + 0.5(2.5)f(2.5) + 1.5(4)f(4)]$

$\approx 2\pi[2 + 12 + 12.5 + 108] = 269\pi$

Sample 1B

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Answer QUESTION 1 parts (c) and (d) on this page.

Response for question 1(c)

for [0, 4], f(r) is increasing (as is r)
so r f(r) is increasing.
As a right Riemann sum was used ~~to~~ take the integral of an increasing function, it was an overestimate

Response for question 1(d)

Avg. value of g(r) on [1, 4] $= \frac{1}{4-1}\int_1^4 (2-16(\cos(1.57\sqrt{r}))^3)\,dr = 9.875794$87

$2-16(\cos(1.57\sqrt{K}))^3 = 9.875794868$

at $K = 2.497$ centimeters

Sample 1C

Answer QUESTION 1 parts (a) and (b) on this page.

r (centimeters)	0	1	2	2.5	4
$f(r)$ (milligrams per square centimeter)	1	2	6	10	18

Response for question 1(a)

$f'(2.25) = \frac{f(2.5)-f(2)}{2.5-2} \rightarrow \frac{10-6}{2.5-2} = \frac{4}{0.5} = 8$

$f'(2.25) = 8$ mg per cm² per cm²

At $r = 2.25$, the rate of the density of the bacteria population is increasing at a rate of 8 mg per cm² per cm²

Response for question 1(b)

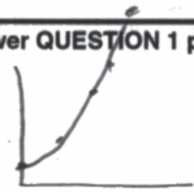
$2\pi \cdot [1(2)+1(6)+0.5(10)+1.5(18)]$

$2\pi \cdot (2+6+5+27)$

$2\pi(40) = \boxed{80\pi = 251.327}$

Answer QUESTION 1 parts (c) and (d) on this page.

Response for question 1(c)



The right Riemann sum approximation is an overestimate due to the fact that the total mass of the bacteria is increasing since its represented by f(r) which is the function of the bacteria's density which is an increasing differentiable function

Response for question 1(d)

$g(r) = 2 - 16(\cos(1.57\sqrt{r}))^3$

$$\frac{1}{4-1}\int_1^4 g(r)\,dr = 4{,}875794868$$

$$g'(k) = 4.875794868$$

Question 1

Note: Student samples are quoted verbatim and may contain spelling and grammatical errors.

Overview

The context of this problem is bacteria in a circular petri dish. The increasing, differentiable function f gives the density of the bacteria population (in milligrams per square centimeter) at a distance r centimeters from the center of the dish. Selected values of $f(r)$ are provided in a table.

In part (a) students were asked to use the table to estimate $f'(2.25)$ and interpret the meaning of this value in context, using correct units. A correct response should estimate the derivative value using a difference quotient, drawing from the data in the table that most tightly bounds $r = 2.25$. The interpretation should explain that when $r = 2.25$ centimeters from the center of the petri dish, the density of the bacteria population is increasing at a rate of roughly 8 milligrams per square centimeter per centimeter.

In part (b) students were told that $2\pi\int_0^4 r f(r)\,dr$ gives the total mass, in milligrams, of the bacteria in the petri dish. They were asked to estimate the value of this integral using a right Riemann sum with the values given in a table. A correct response should multiply the sum of the four products $r_i \cdot f(r_i) \cdot \Delta r_i$ drawn from the table by 2π.

In part (c) students were asked to explain whether the right Riemann sum approximation found in part (b) was an overestimate or an underestimate of the total mass of bacteria. A correct response should determine the derivative of $r \cdot f(r)$ using the product rule, use the given information that f is nonnegative to conclude that this derivative is positive and, therefore, that the integrand is strictly increasing on the interval $0 \le r \le 4$. This means that the right Riemann sum approximation is an overestimate.

In part (d) another function, $g(r) = 2 - 16\left(\cos\left(1.57\sqrt{r}\right)\right)^3$, was introduced as a function that models the density of the bacteria in the petri dish for $1 \le r \le 4$. Students were asked to find the value of k such that $g(k)$ is equal to the average value of $g(r)$ on the interval $1 \le r \le 4$. A correct response should set up the average value of $g(r)$ as $\frac{1}{3}\int_1^4 g(r)\,dr$, then use a graphing calculator to solve for k when setting $g(k)$ equal to this average value.

Sample: 1A
Score: 9

The response earned 9 points: 2 points in part (a), 2 points in part (b), 2 points in part (c), and 3 points in part (d). In part (a) the difference quotient of $\frac{10-6}{0.5}$ in the first line would have earned the first point with no simplification. In this case, correct simplification to 8 in the first line earned the first point. The response earned the second point for an interpretation of the density of the bacteria changing at 8 milligrams per square centimeter per centimeter at $r = 2.25$. In part (b) the response earned the first point for the sum of products expression $2\pi \cdot (2 \cdot 1 \cdot 1 + 6 \cdot 1 \cdot 2 + 10 \cdot 0.5 \cdot 2.5 + 18 \cdot 1.5 \cdot 4)$ in the first line on the right. This sum of products expression would also have earned the second point with no simplification. In this case, correct simplification to 269π in the second line earned the second point. In part (c) the response earned the first point for the product rule expression of $r' \cdot f(r) + r \cdot f'(r)$ for $\frac{d}{dr}(r f(r))$ in the fourth line. The response earned the second point for the conclusion that $r f(r)$ has a positive slope because $r, r', f(r),$ and $f'(r)$ are positive on the interval and, therefore, the estimate is an overestimate. In part (d) the response earned the first and second points for the definite integral $\frac{1}{4-1}\int_1^4 \left(2 - 16\left(\cos\left(1.57\sqrt{r}\right)\right)^3\right) dr$ giving the average value in the first line. The response earned the third point for the correct value of $k = 2.497$ in the third line.

Question 1 (continued)

Sample: 1B
Score: 7

The response earned 7 points: 1 point in part (a), 2 points in part (b), 1 point in part (c), and 3 points in part (d). In part (a) the difference quotient of $\frac{10-6}{0.5}$ in the first line would have earned the first point with no simplification. In this case, correct simplification to 8 in the first line earned the first point. The response did not earn the second point because the density of the bacteria is not referenced. In part (b) the response earned the first point for the sum of products expression $2\pi[1(1)f(1) + 1(2)f(2) + 0.5(2.5)f(2.5) + 1.5(4)f(4)]$ in the first line. The sum of products expression $2\pi[2 + 12 + 12.5 + 108]$ in the second line would have earned the second point with no simplification. In this case, simplification to 269π in the second line earned the second point. In part (c) the response did not earn the first point because there is no product rule expression for $\frac{d}{dr}(rf(r))$. The response earned the second point for the claim that $rf(r)$ is increasing in the second line and, therefore, the right Riemann sum is an overestimate in the fifth line. In part (d) the response earned the first and second points for the definite integral $\frac{1}{4-1}\int_1^4 \left(2 - 16\left(\cos\left(1.57\sqrt{r}\right)\right)^3\right) dr$ giving the average value in the first line. The response earned the third point for the correct value of $k = 2.497$ in the third line.

Sample: 1C
Score: 4

The response earned 4 points: 1 point in part (a), 1 point in part (b), no points in part (c), and 2 points in part (d). In part (a) the difference quotient of $\frac{10-6}{2.5-2}$ in the first line would have earned the first point with no simplification. In this case, correct simplification to 8 in the first line earned the first point. The response did not earn the second point because the response states the rate of the density of the bacteria population is increasing at a rate and because the units in the interpretation are incorrect. In part (b) the response earned one of the two points for providing a completely correct right Riemann sum for $2\pi\int_0^4 f(r)\,dr$. The sum of products expression $2\pi \cdot [1(2) + 1(6) + 0.5(10) + 1.5(18)]$ in the first line would have earned one of the two points with no simplification. In this case, correct simplification to 251.327 in the third line earned one of the two points. In part (c) the response did not earn either point because the response attempted to explain a right Riemann sum for $2\pi\int_0^4 f(r)\,dr$. In part (d) the response earned the first and second points for the definite integral $\frac{1}{4-1}\int_1^4 g(r)\,dr$ giving the average value in the second line. The response did not earn the third point because no value is given for k.